Linear representations of groups with translation invariant defining relationships. Some new series of braid group representations and new invariants of links and knots

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In this paper we indicate one method of construction of linear representations of groups and algebras with translation invariant (except, maybe , finite number) defining relationships. As an illustration of this method, we give one approach to the construction of linear representations of braid group and derive some series of such representations. Some invariants of oriented knots and links are constructed. The author is grateful to Yuri Drozd, Sergey Ovsienko and other members of algebraic seminar at Kiev University for the creative atmosphere without which this work could hardly appear.

1 The description of the approach on the example of the braid group

1.1. **Definition.** *Let N be the set of the natural numbers. Braid group $B_\infty(B_m)$ is the group with generators t_i, $i \in N$ $(i \in \{1, ..., m-1\})$ and the following defining relations:*

(i) $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$,

(ii) $t_i t_j = t_j t_i$ *when* $|j - i| > 1$.

1.2. For natural $n, \lambda, \lambda < n$ let $M_1 = \{1, 2, ..., n\} \subset N$ and for any $i \in N$ let the set M_{i+1} to be obtained of the set M_i by the right translation on the number $n - \lambda$. It is obvious that $|M_i| = n$ and $|M_i \cap M_{i+1}| = \lambda$.

We shall call linear operator $f : V \to V$ on the vector space V over the field k nondegenerate or invertible if there exists linear operator $g : V \to V$ such that $fg = gf = I_V$. For the family $\{V_s\}_{s \in N}$ of the

vector spaces over the field k let $W = \underset{s\in N}{\oplus} V_s$ and $V_M = \underset{s\in M}{\oplus} V_s$ for any $M \subset N$. Let for any $i \in N$ there is some nondegenerate linear operator T_i on the vector space V_{M_i}. Operator T_i is defined by the matrix $\{T_i(s_1+1-\min M_i,\ s_2+1-\min M_i)\}_{\substack{s_1\in M_i\\ s_2\in M_i}}$, where $T_i(s_1+1-\min M_i, s_2+1-\min M_i)$ is the linear operator from the vector space V_{s_2} into the vector space V_{s_1}. We shall extend this operator to the linear operator $\tilde{t}_i$ on W by setting $\tilde{t}_i = (T_i)_{V_{M_i}} \oplus 1_{V_{N\setminus M_i}}$. Relations 1.1. (i)–(ii) that hold if and only if they hold on the vector spaces $V_{M_i\cup M_{i+1}}$ (for 1.1. (i)) and $V_{M_i\cup M_j}$ (for 1.1. (ii)), define some relations for the matrix elements of matrices T_i. If they are true we get linear representation of the braid group B_∞. We note that if $\lambda \le \frac{n}{2}$ relations 1.1. (ii) hold automatically.

Remark. *If there exists $d \in N$ such that for any i $T_i = T_{i+d}$ in some basises in $V_s, s \in N$ (that is if we consider periodic case with period d), then we have already the finite number of matrices $T_1, ..., T_d$ with finite number of relations for their matrix elements (and thus, some "algebra with finite number of generators and defining relations") such that if they hold we obtain a linear representation of the group $\mathcal{B}_\infty$. The same approach is true to any group (and some rings) that can be defined by translation invariant relationships. It is the crux of this article.*

1 The same approach is true if we consider instead of direct sum tensor product. Namely, for the braid group B_m let us consider

$$W = \otimes \prod_{s=1}^{s=\lambda+(m-1)(n-\lambda)} V_s, \qquad V_M = \otimes \prod_{s\in M} V_s,$$

where $M \subset N$. Let $T(i)$, for any $i \in \{1, ..., m-1\}$ be some nondegenerate linear operator on the vector space V_{M_i}. Thus $T(i)$ is defined by tensor $T(i) \in V_{M_i} \otimes V^*_{M_i}$. We shall define the operator $\tilde{t}_i$ on vector space W by setting $\tilde{t}_i = T(i)_{V_{M_i}} \otimes 1_{V_{\{1,...,\lambda+(m-1)(n-\lambda)\}\setminus M_i}}$.

Relations 1.1. (i)-(ii) that hold if and only if they hold on the vector space $V_{M_i\cup M_{i+1}}$ (for 1.1 (i)) and $V_{M_i\cup V_{M_j}}$ (for 1.1 (ii)) define some relations for the $T(i)$. If $\lambda \le \frac{n}{2}$ relations 1.1 (ii) hold automatically. The obvious analog of the remark above is true for the case of tensor products (that is we separate periodic case, when for some $d \in N$ $T(i+d) = T(i)$

for any i and, thus, all relations for $T(i)$ are consequences of the finite number of relations for $T(1), ..., T(d)$).

2 Linear representations of the braid group

To illusrate as told above we consider at first the simplest case $(n, \lambda) = (2, 1)$.

2.1. Relations 1.1. (ii) hold evidently. On the vector space $V_{M_i \cup M_{i+1}} = V_i \oplus V_{i+1} \oplus V_{i+2}$ matrices of the linear operators $\tilde{t}_i, \tilde{t}_{i+1}$ have the following form:

$$T_i = \begin{pmatrix} A_i & B_i & 0 \\ C_i & D_i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_{i+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_{i+1} & B_{i+1} \\ 0 & C_{i+1} & D_{i+1} \end{pmatrix}.$$

The condition $\tilde{t}_i \tilde{t}_{i+1} \tilde{t}_i = \tilde{t}_{i+1} \tilde{t}_i \tilde{t}_{i+1}$, $i \in N$ holds then and only then when $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$. Simple calculation shows that it is equivalent to the following relations:

(i) $A_i^2 + B_i A_{i+1} C_i = A_i$,
(ii) $D_{i+1}^2 + C_{i+1} D_i B_{i+1} = D_{i+1}$,
(iii) $A_{i+1} C_i - C_i A_i = D_i A_{i+1} C_i$,
(iv) $B_i A_{i+1} - A_i B_i = B_i A_{i+1} D_i$,
(v) $C_i B_i - B_{i+1} C_{i+1} = A_{i+1} D_i A_{i+1} - D_i A_{i+1} D_i$,
(vi) $C_{i+1} D_i - D_{i+1} C_{i+1} = C_{i+1} D_i A_{i+1}$,
(vii) $D_i B_{i+1} - B_{i+1} D_{i+1} = A_{i+1} D_i B_{i+1}$

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Let us consider the case of tensor product. Relations 1.1 (ii) hold evidently. Simple calculation shows that relations $\tilde{t}_i \tilde{t}_{i+1} \tilde{t}_i = \tilde{t}_{i+1} \tilde{t}_i \tilde{t}_{i+1}$ are equivalent to the following relations for the tensors $T(i) \in (V_i \otimes V_{i+1}) \otimes (V_i \otimes V_{i+1})^*$, $i = 1, ..., m-1$:

$$\text{(viii)} \sum_{k_1,k_2,k_3} T(i)^{i_1 i_2}{}_{k_1 k_3} \times T(i+1)^{k_3 i_3}{}_{k_2 j_3} \times T(i)^{k_1 k_2}{}_{j_1 j_2} =$$

$$= \sum_{k_1,k_2,k_3} T(i+1)^{i_2 i_3}{}_{k_1 k_3} \times T(i)^{i_1 k_1}{}_{j_1 k_2} \times T(i+1)^{k_2 k_3}{}_{j_2 j_3}$$

for any $i_1, i_2, i_3, j_1, j_2, j_3$, $1 \leq i_1(j_1) \leq n_i$, $1 \leq i_2(j_2) \leq n_{i+1}$, $1 \leq i_3(j_3) \leq n_{i+2}$, where $n_i = \dim V_i$. Here elements $T(i)^{uv}{}_{pq}$ of the field k

are defined by some choice of the basis $\{e_s^{(i)}\}_{s=1,\dots,n_i}$ in every V_i and by the equality

$$T(i) = \sum T(i)^{uv}{}_{pq} e_u^{(i)} \otimes e_v^{(i+1)} \otimes e^{(i)p*} \otimes e^{(i+1)q*}.$$

Thus we have the following theorem.

Theorem. *Let linear operator T_i on the vector space $V_i \oplus V_{i+1}$ with matrix $\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ is nondegenerate for any $i \in N$ and relations* (i)-(vii) *hold. Then linear operators T_i define representation of the braid group B_∞.*

Let $T(i) \in (V_i \otimes V_{i+1}) \otimes (V_i \otimes V_{i+1})^*$ be such linear operator on the vector space $V_i \otimes V_{i+1}$ that matrix $(T(i)^{uv}{}_{pq})$is nondegenerate for any $i \in \{1, ..., m-1\}$ and relations (viii) *hold. Then linear operators T_i define representation of the braid group B_m.*

Now let us come to the periodic case according to the approach pointed out in remark 1.2.

2.2. Let $\mathcal{B}$ be an algebra over field k with generators A, B, C, D and the following defining relations:

(i) $A^2 + BAC = A$,
(ii) $D^2 + CDB = D$,
(iii) $CB - BC = ADA - DAD$,
(iv) $BA - AB = BAD$,
(v) $AC - CA = DAC$,
(vi) $DB - BD = ADB$,
(vii) $CD - DC = CDA$.

3 Let $T \in V^{\otimes 2} \otimes (V^*)^{\otimes 2}$ be the tensor of type (2.2), such that the following relation holds:

$$\text{(viii)} \sum_{k_1,k_2,k_3} T^{i_1 i_2}{}_{k_1 k_3} \times T^{k_3 i_3}{}_{k_2 j_3} \times T^{k_1 k_2}{}_{j_1 j_2} =$$

$$= \sum_{k_1,k_2,k_3} T^{i_2 i_3}{}_{k_1 k_3} \times T^{i_1 k_1}{}_{j_1 k_2} \times T^{k_2 k_3}{}_{j_2 j_3}$$

for any $i_1, i_2, i_3, j_1, j_2, j_3$, $1 \le i_1(j_1) \le \dim V$, $1 \le i_2(j_2) \le \dim V$, $1 \le i_3(j_3) \le \dim V$.

Theorem. *Any representation π of algebra $\mathcal{B}$ $\pi : \mathcal{B} \to \mathrm{End}_k(V)$ over field k, such that linear operator $t : V \oplus V \to V \oplus V$ with matrix $T = \begin{pmatrix} \pi(A) & \pi(B) \\ \pi(C) & \pi(D) \end{pmatrix}$ is nondegenerate, defines the representation of the braid group B_∞ over the field k. 4 Any tensor $T \in V^{\otimes 2} \otimes (V^*)^{\otimes 2}$, such that relation* (viii) *holds and matrix $(T^{uv}{}_{pq})$ is nondegenerate defines the representation of the braid group B_m, for any m.*

The proof of the theorem is obvious if we set $V_i \simeq V$ for any $i \in N$. In the case of theorem $d = 1$ because $T_{i+1} = T_i$ (see remark 1.2.).

Definition. *We shall call an algebra $\mathcal{B}$ the braid algebra 5 ot type (2,1) of period 1 (or simply the braid algebra). We shall call relation* (viii) *the braid equation of type (2,1) of period 1 (or simply the braid equation).*

Note that some decisions of the braid equation can be obtained via quantum groups and Yang-Baxter equation, see [Dr], [T], [K].

2.3. Let V_1, V_2 be two vector spaces over the field k, $A_1, D_2 : V_1 \to V_1$, $B_1, C_2 : V_2 \to V_1$, $C_1, B_2 : V_1 \to V_2$, $D_1, A_2 : V_2 \to V_2$ linear operators such that following relations hold:

$$\begin{array}{llll}
\text{(i)} & A_1^2 + B_1A_2C_1 = A_1; & \text{(vii)} & A_2C_1 - C_1A_1 = D_1A_2C_1; \\
\text{(ii)} & A_2^2 + B_2A_1C_2 = A_2, & \text{(viii)} & A_1C_2 - C_2A_2 = D_2A_1C_2, \\
\text{(iii)} & D_2^2 + C_2D_1B_2 = D_2; & \text{(ix)} & B_1A_2 - A_1B_1 = B_1A_2D_1; \\
\text{(iv)} & D_1^2 + C_1D_2B_1 = D_1, & \text{(x)} & B_2A_1 - A_2B_2 = B_2A_1D_2, \\
\text{(v)} & D_1B_2 - B_2D_2 = A_2D_1B_2; & \text{(xi)} & C_2D_1 - D_2C_2 = C_2D_1A_2; \\
\text{(vi)} & C_1D_2 - D_1C_1 = C_1D_2A_1, & \text{(xii)} & D_2B_1 - B_1D_1 = A_1D_2B_1.
\end{array}$$

$$\begin{array}{ll}
\text{(viii)} & C_1B_1 - B_2C_2 = A_2D_1A_2 - D_1A_2D_1; \\
\text{(xiv)} & C_2B_2 - B_1C_1 = A_1D_2A_1 - D_2A_1D_2,
\end{array}$$

Let $T(1) \in (V_1 \otimes V_2) \otimes (V_1 \otimes V_2)^*$, $T(2) \in (V_2 \otimes V_1) \otimes (V_2 \otimes V_1)^*$ be two tensors, such that for any $i_1, i_2, i_3, j_1, j_2, j_3$, $1 \le i_1(j_1) \le \dim V_1$, $1 \le i_2(j_2) \le \dim V_2$, $1 \le i_3(j_3) \le \dim V_1$, the following relations hold:

$$\text{(xv)} \sum_{k_1,k_2,k_3} T(1)^{i_1i_2}{}_{k_1k_3} \times T(2)^{k_3i_3}{}_{k_2j_3} \times T(1)^{k_1k_2}{}_{j_1j_2} =$$

$$= \sum_{k_1,k_2,k_3} T(2)^{i_2i_3}{}_{k_1k_3} \times T(1)^{i_1k_1}{}_{j_1k_2} \times T(2)^{k_2k_3}{}_{j_2j_3}$$

$(1 \le k_1 \le \dim V_1, \quad 1 \le k_2 \le \dim V_2, \quad (1 \le k_3 \le \dim V_1;)$

$$\text{(xvi)} \sum_{k_1,k_2,k_3} T(2)^{i_2 i_3}{}_{k_1 k_3} \times T(1)^{k_3 i_1}{}_{k_2 j_1} \times T(2)^{k_1 k_2}{}_{j_2 j_3} =$$

$$= \sum_{k_1,k_2,k_3} T(1)^{i_3 i_1}{}_{k_1 k_3} \times T(2)^{i_2 k_1}{}_{j_2 k_2} \times T(1)^{k_2 k_3}{}_{j_3 j_1}$$

$(1 \le k_1 \le \dim V_2, \quad 1 \le k_2 \le \dim V_1, \quad (1 \le k_3 \le \dim V_2;)$

Theorem. *If linear operators* $T_1 : V_1 \oplus V_2 \to V_1 \oplus V_2$ *with matrix* $T_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$, $T_2 : V_2 \oplus V_1 \to V_2 \oplus V_1$ *with matrix* $T_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ *are nondegenerate, and relations* (i)-(xiv) *hold then linear operators* $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ *define representation of the braid group* B_∞.

If matrices $(T(1)^{i_1 i_2}{}_{j_1 j_2})$, $(T(2)^{i_1 i_2}{}_{j_1 j_2})$ *are nondegenerate and relations* (xv)-(xvi) *hold then tensors* $T(1)$ *and* $T(2)$ *define representation of the braid group* B_m, *for any* $m \in N$.

The proof of the theorem is obvious if we set $V_i = V_1$ for odd i and $V_i = V_2$ for even i.

In the case of the theorem $d = 2$, because $T_{i+2} = T_i$ (see remark 1.2.). It is obvious that analogous theorem can be formulated for any d. We should call the relations (i)-(xix) the braid algebra of type (2,1) of period 2 and relations (xv)-(xvi) the braid equation of type (2,1) of period 2.

2.4. **Definition.** *Let* $k[x, x^{-1}]$ *be the ring of Laurent polynomials over field* k *and* $\mathcal{B}^{\triangle} = k[x, x^{-1}][y]$ *is the commutative algebra with the relation* $y^2 + xy = y$. *We shall call this algebra triangle braid algebra.*

Theorem. *Suppose* $\tilde{t}_i, i \in N$ *(see 1.2 and 2.1) satisfies the conditions of the theorem 2.1 and* $A_i = 0$ $(D_i = 0)$ *for any* $i \in N$. *Then* $V_i \simeq V_j \simeq V$ *for any* $i, j \in N$ *and some vector space* V, $W \simeq \bigoplus_{s \in N} V$ *and the representation* π, *defined according to the theorem 2.1, is equivalent to the representation* π' *with* $A'_i = 0 (D'_i = 0)$, $C'_i = 1_V$, $B'_i = B$, $D'_i = D(A'_i = A)$ *for any* i, *where* $D(A) : V \to V$, $B : V \to V$

linear operators, and B is nondegenerate (in particular, it is the case of theorem 2.2.). There exists one–to–one correspondence between such representations π' of the braid group B_∞ and the representations of the algebra $\mathcal{B}^\triangle$.

Proof. Matrix of linear operator $\tilde{t}_i$ on the vector space $V_i \oplus V_{i+1}$ is nondegenerate and it has the triangle form $\begin{pmatrix} 0 & B_i \\ C_i & D_i \end{pmatrix} \left(\begin{pmatrix} A_i & B_i \\ C_i & 0 \end{pmatrix} \right)$.

Hence linear operators C_i, B_i are nondegenerate and $V_i \simeq V_{i+1}$. We can choose basis in every $V_i, i \in N$ such that matrix for any $C_i, i \in N$ is I. After substituting $C_i = I, A_i = 0 (D_i = 0)$ into the relations 2.1. (i)–(vii) they come to the following form:

$$\begin{array}{ll} BD = DB & (BA = AB), \\ D^2 + BD = D & (A^2 + AB = A). \end{array}$$

Taking to account that B must to be nondegenerate linear operator, we obtain the statement of the theorem $(y \to D(A), x \to B)$.

Remark. *If the representation of the algebra $\mathcal{B}^\triangle$ is direct sum of some representations then corresponding representation of the group B_∞ is the direct sum of the corresponding representations of the braid group B_∞.*

The equation $y^2 + xy = y$ is equivalent to the equation $y(y + x - 1) = 0$. So it has at least two obvious solutions $y = 0$ and $y = 1 - x$. Thus for any nondegenerate linear operator B on the vector space V over field k we have two representations $f_1 : \mathcal{B}^\triangle \to \mathrm{End}_k V$ and $f_2 : \mathcal{B}^\triangle \to \mathrm{End}_k V$, where $f_1(x) = B,\ f_1(y) = 0,\ f_2(x) = B,\ f_2(y) = I - B$. So we have following cosequence of the theorem (we also took into account the remark above) that gives three series I, II, III of the representations of the braid group B_∞:

Consequence. Let B be linear nondegenerate indecomposable operator $V \to V$ for some vector space V (so if k is algebraically closed field and $\dim V < \infty$, then B is Jordan box over field k with nonzero eigenvalue). There are representations of the items 1.2. and 2.1. with:

(i) (seriesI)$A_i = 0, D_i = 0, C_i = I, B_i = B$ for any $i \in N$;

(ii) (seriesII)$A_i = 0, C_i = I, B_i = B, D_i = I - B$ for any $i \in N$;

(iii) (seriesIII)$A_i = 1 - B, B_i = B, C_i = I, D_i = 0$ for any $i \in N$;

2.5. There are other more veiled representations of the commutative algebra $\mathcal{B}^{\triangle}$.

Theorem. *There is one-to-one correspondence between finite-dimensional indecomposable representations over field k of the commutative algebra $\mathcal{B}^{\triangle}$, different from representations f_1 and f_2 (see 2.4.), and the finite-dimensional indecomposable representations over the field k of the ordered pairs linear operators (π_1, π_2) with relations $\pi_1\pi_2 = 0$ and $\pi_2\pi_1 = 0$ such that $\pi_1 \neq 0, \pi_2 \neq 0$ (corresponding algebra is, obviously, commutative algebra $k[x, y]$ with the only relation $xy = 0$).*

Proof. Let $f : \mathcal{B}^{\triangle} \to \mathrm{End}_k V$ be indecomposable representation of the algebra $B^{\triangle}$, $f \neq f_1, f \neq f_2$. Then $f(y)(f(y) + f(x) - I) = (f(y) + f(x) - I)f(y) = 0$. Let $\pi_1 = f(y), \pi_2 = f(y) + f(x) - I$. Then $\pi_1\pi_2 = \pi_2\pi_1 = 0$, $\pi_1 \neq 0, \pi_2 \neq 0$ and it easy to verify that this representation of the pair of linear operators is indecomposable. Conversely, let (π_1, π_2) be the indecomposable representation of the pair of linear operators in vector space V, $\dim V < \infty$ such that $\pi_1\pi_2 = \pi_2\pi_1 = 0$, $\pi_1 \neq 0, \pi_2 \neq 0$. If $\pi_1(\pi_2)$ is not nilpotent operator then $\pi_2 = 0(\pi_1 = 0)$, see [NRSB]. Hence π_1 and π_2 are nilpotent operators. Put $f(y) = \pi_1, f(x) = \pi_2 - \pi_1 + I$. As π_2 and π_1 commutate one with another, linear operator $\pi_2 - \pi_1$ is nilpotent too and hence linear operator $f(x) = \pi_2 - \pi_1 + I$ is invertible. Theorem is proved.

Consequence. *Every finite dimensional indecomposable representation of the pair operators π_1, π_2 with relations $\pi_1\pi_2 = \pi_2\pi_1 = 0$ such that $\pi_1 \neq 0, \pi_2 \neq 0$ defines representation of the braid group B_∞.*

Proof. It follows from the theorem and the theorem 2.4.

2.6. All indecomposable representations of the pair operators π_1, π_2 such that $\pi_1\pi_2 = \pi_2\pi_1 = 0, \pi_1 \neq 0, \pi_2 \neq 0$ have been found in [GP] for

the case, when k is algebraically closed field, and in [NRSB] for the case of arbitrary field k.

Theorem [NRSB]. Every indecomposable finite-dimensional representation of the pair operators π_1, π_2 with relations $\pi_1\pi_2 = \pi_2\pi_1 = 0, \pi_1 \neq 0, \pi_2 \neq 0$ has one of the two types:

(i) vector space V of type I is defined by sequence of pairs not negative integers $(k_1, l_1), ..., (k_n, l_n)$, where $k_\alpha > 0$ when $\alpha > 1, l_\beta > 0$ when $\beta < n$. Vector space V is built like $1 + \sum(k_\alpha + l_\alpha)$-dimensional space with basis of vectors $v_\alpha\pi_1^k, v_\alpha\pi_2^l, v_n\pi_2^{l_n}$, where $k = 0, ..., k_\alpha, l = 1, ..., l_\alpha - 1$, $\alpha = 1, ..., n$. Equality $\pi_1\pi_2 = \pi_2\pi_1 = 0$, form of the basis vectors and relations

$$v_1\pi_1^{k_1+1} = 0,\ v_{\alpha+1}\pi_1^{k_{\alpha+1}} = v_\alpha\pi_2^{l_\alpha}\ (\alpha = 1, ..., n-1),\quad v_n\pi_2^{l_n+1} = 0$$

completely define the action of operators π_1 and π_2 on the vectors of the basis;

(ii) vector space V of type II is defined by

a) sequence $(r_1, s_1), ..., (r_q, s_q)$ of pairs of natural numbers that is nonperiodic . It means that if this sequence coincide with sequence $(r_1, s_1,), ..., (r_t, s_t), (r_1, s_1), ..., (r_t, s_t)$ then $t = q$.

Sequences that differ on cyclic substitution are equivalent,

b) polynomial $f(x) = x^t - b_1x^{t-1} - \cdots - b_t(b_t \neq 0)$ over the field k that is the power of the indecomposable polynomial over the field k.

Vector space V of type II has basis of the vectors $v_{\mu\nu}\pi_1^r, v_{\mu\nu}\pi_2^s$, where $r = 0, ..., r_\mu$, $s = 1, ..., s_{\mu-1}$, $\mu = 1, ..., q$, $\nu = 1, ..., t$. Operators π_1 and π_2 are completely defined by following relations:

$$\begin{aligned}
&v_{11}\pi_1^{r_1} = (v_{q1}b_1 + \cdots + v_{qt}b_t)\pi_2^{s_q},\\
&v_{1,\nu+1}\pi_1^{r_1} = v_{q\nu}\pi_2^{s_q}\ (\nu = 1, ..., t-1),\\
&v_{\mu+1,\nu}\pi_1^{r_{\mu+1}} = v_{\mu\nu}\pi_2^{s_\mu}\ (\mu = 1, ..., q-1, \nu = 1, ..., t-1).
\end{aligned}$$

Consequence. *There are series IV and V of the representations of the braid group B_∞ originating from the representations of pair operators (π_1, π_2) of type I and II, respectively.*

Proof. It follows from the consequence 2.5. and the theorem.

Theorem 2.5 and the theorem [NRSB] give the complete description of the finite-dimensional representations of the triangle braid algebra $\mathcal{B}^\triangle$.

In [NRSB] authors note the fact that the description of Lorentz group representations and the classification of finite p-groups with abelian subgroup of index p are connected with the same problem of description mutually annihilating operators is remarkable. Now we see that we have to mention that this problem is connected with braid group representations too.

2.7. **Theorem.** *Let $\pi : \mathcal{B} \to \mathrm{End}_k V$ be the representation of the braid algebra $\mathcal{B}$ over field k, such that $\pi(C) = I$. Then the corresponding representation of the braid group B_∞ (see theorem 2.2.), in such case, when operator on the vector space $V \oplus V$ with the matrix $\begin{pmatrix} \pi(A) & \pi(B) \\ I & \pi(D) \end{pmatrix}$ is nondegenerate, may be obtained of some representation π' of algebra $\widetilde{\mathcal{B}}$ over field k with generators A, B, D and the following defining relations:*

(i) $A^2 + BA = A$,
(ii) $D^2 + DB = D$,
(iii) $BA - AB = BAD$,
(iv) $DB - BD = ADB$,
(v) $DA = 0$.

Conversely, any representation π of algebra $\widetilde{\mathcal{B}}$ $\pi : \mathcal{B} \to \mathrm{End}_k V$, such that operator on vector space V with matrix $\begin{pmatrix} \pi(A) & \pi(B) \\ I & \pi(D) \end{pmatrix}$ is nondegenerate, gives the representation of the braid group B_∞.

Proof. It is obvious.

Definition. We shall call the algebra $\widetilde{\mathcal{B}}$ simplified braid algebra.

2.8. **Theorem.** *Quotient algebra of the braid algebra $\mathcal{B}$ on it commutant is isomorphic to the commutative algebra $\mathcal{B}^{\mathrm{com}} = k[x, y, z, t]$ with following difining relations:*

(i) $x^2 + xyz = x$,
(ii) $t^2 + tyz = t$,
(iii) $xzt = 0$,
(iv) $xyt = 0$,

Any representation π of the algebra $\mathcal{B}^{\mathrm{com}}$ $\pi : \mathcal{B}^{\mathrm{com}} \to \mathrm{End}_k V$, over the field k such that operator on the vector space $V \oplus V$ with matrix

$\begin{pmatrix} \pi(x) & \pi(y) \\ \pi(z) & \pi(t) \end{pmatrix}$ *is nondegenerate defines the representation of the braid group* B_∞.

Proof. It is obvious (here $A \to x, B \to y, C \to z, D \to t$).

Definition. We shall call algebra $\mathcal{B}^{\mathrm{com}}$ commutative braid algebra.

Remark. We note that if the representation π of the braid group B_∞ is obtained from some representation of the commutative braid algebra, it doesn´t mean, of course, that $\pi(B_\infty)$ is commutative.

2.9. **Theorem.** *Quotient algebra of the simplified braid algebra* $\widetilde{\mathcal{B}}$ *on its commutant is isomorphic to the commutative algebra* $\widetilde{\mathcal{B}}^{\mathrm{com}} = k[x, y, t]$ *with the following defining relations:*

(i) $x^2 + xy = x$,
(ii) $t^2 + ty = t$,
(iii) $xt = 0$.

Any representation π *of the algebra* $\widetilde{\mathcal{B}}^{\mathrm{com}}$ $\pi : \widetilde{\mathcal{B}}^{\mathrm{com}} \to \mathrm{End}_k V$,
in the vector space V *over the field* k*, such that operator on the vector space* $V \oplus V$ *with matrix* $\begin{pmatrix} \pi(x) & \pi(y) \\ I & \pi(t) \end{pmatrix}$ *is nondegenerate, defines the representation of the braid group* B_∞.

Proof. It is obvious (here $A \to x, B \to y, D \to t$).

Definition. We shall call algebra $\widetilde{\mathcal{B}}^{\mathrm{com}}$ simplified commutative braid algebra.

Remark. As items 2.5 and 2.7 show any representation of the braid group B_∞, obtained of some finite-dimensional representation of algebra $\mathcal{B}^{\mathrm{com}}$, may be obtained from some finite-dimensional representation of algebra $\widetilde{\mathcal{B}}^{\mathrm{com}}$.

2.10. **Remark.** If the representation of the algebra $\widetilde{\mathcal{B}}^{\mathrm{com}}$ is direct sum of some representations, then corresponding representation of the braid group B_∞ is the direct sum of the corresponding representations of the braid group B_∞.

Let R_1 be the quotient algebra of algebra $\widetilde{\mathcal{B}}^{\mathrm{com}}$ on the relation $t = \alpha x, \alpha \in k, \alpha \neq 0$. Then R_1 is commutative algebra $k[x, y]$ with the following relations:

(i) $x^2 = 0$,
(ii) $x(y-1) = 0$.

The relation (ii) holds automatically if $y = 1 + \beta x$, for any $\beta \in k$. So we have the commutative algebra $R_2 = k[x]$ with the only relation $x^2 = 0$. Its indecomposable representation is defined by the Jordan 2×2 box with zero eigenvalue $x \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. So we have representation $\pi_{\alpha,\beta}$ of algebra $\widetilde{\mathcal{B}}^{\mathrm{com}}$ in the vector space V with basis e_1, e_2 such that $\pi_{\alpha,\beta}(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\pi_{\alpha,\beta}(y) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$, $\pi_{\alpha,\beta}(t) = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$. It is easy to verify that $\pi_{\alpha,\beta}$ is indecomposable. Operator on the vector space $V \oplus V$ with matrix $\begin{pmatrix} \pi_{\alpha,\beta}(x) & \pi_{\alpha,\beta}(y) \\ I & \pi_{\alpha,\beta}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \beta \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 \end{pmatrix}$ is nondegenerate, hence, according to theorem 2.9, $\pi_{\alpha,\beta}$ defines the representation of the braid group B_∞. Matrices of the operators $\tilde{t}_i, \tilde{t}_{i+1}$ in the vector space $W = \bigoplus_{s \in N} V_s (V_s \simeq V$ for any $s \in N)$ in the basis $e_1^{(s)}, e_2^{(s)}, s = 1, 2, \ldots$ (not indicated matrix elements are 1 if they are diagonal, and 0 if not) have the next form:

$$\tilde{t}_i = \begin{array}{c} \begin{matrix} e_1^{(i)} & e_2^{(i)} & e_1^{(i+1)} & e_2^{(i+1)} & e_1^{(i+2)} & e_2^{(i+2)} \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array};$$

$$\tilde{t}_{i+1} = \begin{array}{c} \begin{matrix} e_1^{(i)} & e_2^{(i)} & e_1^{(i+1)} & e_2^{(i+1)} & e_1^{(i+2)} & e_2^{(i+2)} \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \beta \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}.$$

One can easily verify that $\tilde{t}_i\tilde{t}_{i+1}\tilde{t}_i = \tilde{t}_{i+1}\tilde{t}_i\tilde{t}_{i+1}$. This series of the braid group representations we shall call series VI.

The next theorem generalize the above example of the representation of algebra $\widetilde{\mathcal{B}}^{\mathrm{com}}$.

Theorem. *Let R be a finite-dimensional algebra over the field k and J is an ideal in R, such that $J^2 = 0$. For the elements $a, b, c \in J$ put $\varphi(x) = a$, $\varphi(t) = b$, $\varphi(y) = 1 + c$. Then φ determines homomorphism of algebras over the field k $\varphi : \tilde{\mathcal{B}}^{\mathrm{com}} \to R$. Put $\pi = \psi\varphi$, where ψ is a regular representation of algebra R over the field k. Then π defines finite-dimensional linear representation of algebra $\tilde{\mathcal{B}}^{\mathrm{com}}$, such that matrix $\begin{pmatrix} \pi(x) & \pi(y) \\ I & \pi(t) \end{pmatrix}$ is nondegenerate and, hence, the representation of the group B_∞.*

Proof. It is obvious.

Definition. *The algebra over the field k (the group) is called wild if the problem of classification of its linear representations contains the problem of pair matrices (that is the problem of classification for any pair matrices $A, B \in \mathrm{Mat}_n(k)$, $n > 1$ the orbits XAX^{-1}, XBX^{-1}, $X \in GL_n(k)$), see also [D1].*

Note that if the algebra is wild it has, roughly speaking, "a huge amount" of indecomposible representations.

Consequence. *Algebra $\tilde{\mathcal{B}}^{\mathrm{com}}$ (and, hence, $\mathcal{B}$) is wild.*

Proof. Let

$$a = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}_{2n\times 2n}, b = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}_{2n\times 2n}, c = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}_{2n\times 2n},$$

where $A, B, C \in \mathrm{Mat}_n(k)$, $R = k < a, b, c > \subset \mathrm{Mat}_{2n}(k)$, $J = RaR + RbR + RcR$. Then we appear in the conditions of the theorem. But the problem of classification of the triple of matrices $a, b, c + 1 \in \mathrm{Mat}_{2n}(k)$ is wild, see [D2].

Remark. *The above consequence, it seems, is a very favourable for the theory of links invariants.*

Series I-VI, theorem and the consequence give, in certain sense, the decision of the problem 25 of Joan Birman (Problem 25. Finite representations of the braid groups), see [M].

If k is a finite field we get from the series I-VI and the theorem the finite quotiens of B_n (in fact, most of obtained representations can be defined, when k is a ring, in particular, when k is a finite ring,for instance $k = Z/mZ$, $m \in N$). It gives a lot of decisions of the problem 25.1 (Problem 25.1. Identify explicit interesting finite quotients of B_n), see [M].

Of course, the fact, that we have found simple quotient algebra of the braid algebra $\mathcal{B}$ (namely the commutative braid algebra) mainly contributed to the obtaining of the results of this item. 2.11. There are other interesting quotient algebras of the braid algebra. For instance, if we factorize the braid algebra on the relation $AD = 0$, we obtain noncommutative algebra over the field k with generators A, B, C, D and with the following relations:

$$
\begin{array}{rl}
\text{(i)} & A^2 + BAC = A, \\
\text{(ii)} & D^2 + CDB = D, \\
\text{(iii)} & CB = BC, \\
\text{(iv)} & AB = BA, \\
\text{(v)} & AC - CA = DAC \\
\text{(vi)} & CD - DC = CDA \\
\text{(vii)} & AD = 0.
\end{array}
$$

2.12. We may suppose that there are "mixed" series, "glued together" of obtained series. We mean that possibly there is a representation of the braid group B_∞ such that for some $n_1 \in N$ $\tilde{t_i}$ could be of the series k_1 for $i \in [1, n_1]$, then for some $n_2 > n_1, t_i$, are of the "glue" if $i \in [n_1 + 1, n_2]$, then for some $n_3 > n_2$ $\tilde{t_i}$ could be of the series k_2 if $i \in [n_2 + 1, n_3]$ and so on. It is interesting to investigate if this could happen and if "yes", to describe "all kinds of glue" that could "stick" series k_1 and k_2.

2.13. The theory of the representation of the infinite group G includes the description all nonizomorphic short exact sequences

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

of G modules. In this paper we give only some examples of B_∞ modules, leaving the investigation of their submodules and quotient modules for further publications as well as consideration on the base of the same approach of Briescorn group´s representations, bridge algebra representations and so on (we mean the theory of the representations of algebras and groups with transferable invariant defining relations, see remark 1.2.).

We leave also for further publication functional aspects of the approach of this article, that is, when $k = R$ or $k = C$ and V_i in $W = \bigoplus_{i \in N} V_i$ have the structure of Hilbert or unitar spaces and we consider unitary, self-conjugate operators and so on.

2.14. It is easy to formulate the statement analoguos theorem 2.2 (that is, for $d = 1$) for the case $(n, \lambda) = (3, 1)$; $(n, \lambda) = (3, 2)$; $(n, \lambda) = (4, 1)$; $(n, \lambda) = (4, 2)$ etc. We shall call the corresponding tensor the braid tensor of type (n, λ) and the corresponding algebra the braid algebra of type (n, λ) and denote it $\mathcal{B}_{n,\lambda}$. Thus $\mathcal{B} = \mathcal{B}_{2,1}$. Despite the big number of generators and defining relations, some of the factorizations of these algebras are "not complex". We note that there is an obvious map from the set of representations of algebra $\mathcal{B}_{nm,\lambda m}$ for any $m \in N$ into the set of the representations of algebra $\mathcal{B}_{n,\lambda}$. Some of the representations of these algebras $\mathcal{B}_{n,\lambda}$ (now hypothetical) generate representations of the braid group B_∞ that "detail" those obtained from the representations of the braid algebra $\mathcal{B}_{n_1,\lambda_1}$ (not only in the case when $n = n_1 m, \lambda = \lambda_1 m$ for some $m \in N$) in particular of the braid algebra $\mathcal{B}$. The amount of the possibilites and combinatorics that appear here are impressive.

2.15. "*Irresponsible dynamical system interpretation*".

Suppose that all representations of the braid group B_∞ are the points of the variety (or may be the projective or injective limit of varieties). Map F of this variety (into itself) can be described in the following way: to every representation π (the point of variety) of the braid group B_∞ we set point $F(\pi)$, namely representation $F(\pi)$ such that $F(\pi)(g_i) = \pi(g_{i+1})$ for any $i \in N$. Thus we have "the dynamical system "on the "variety" of all representations of the group B_∞. The obvious analog of the theorem 2.2 for the general case (n, λ) and $d > 1$ gives the description of some periodic orbits of such dynamical system, and the obvious analog of theorem 2.2 for the algebras $\mathrm{B}_{n,\lambda}$ in case $d = 1$ gives the description of

some fixed points of this dynamical system. If the sequence $\{F^n(\pi)\}_{n\geq 1}$ "converges" to some point π', then π' is a fixed point. The interesting question is, can any fixed point (or more widely, can any periodic orbit) of this dynamical system be obtained by the approach utilized in this paper?

2.16. For any $n \in N$ let $B_n \subset B_\infty$ be the group generated by the elements $t_1, ..., t_{n-1}$. It is called the braid group with n strings [B]. Let π be a finite-dimensional representation of the group B_n. Algebra $k < \pi(t_1), ..., \pi(t_{n-1}) >$ generated over the field k by operators $\pi(t_1), ..., \pi(t_{n-1})$ has the finite dimension over the field k. For the group algebra kB_n of B_n over the field k let ideal J_π be the kernel of the map $kB_n \to k < \pi(t_1), ..., \pi(t_{n-1}) > (t_i \to \pi(t_i))$. Let J be any ideal of the algebra kB_n, such that $J \subset J_\pi$ and the quotient algebra $kB_n/J = H_{\pi,J,n}$ has the finite dimension over the field k. Such algebras $H_{\pi,J,n}$ we call "Hecke alike" algebras generated by the representation π. Among the linear representations of such algebra there is one that defines representation π of the braid group B_n, namely $kB_n \to kB_n/J \to kB_n/J_\pi$. Thus, algebras $H_{\pi,J,n}$ "multiply" the representation π. The most interesting is the case when J is minimal ideal in kB_n among those with the finite dimensional quotient algebra. If we have finite-dimensional linear representation π of the group B_n it is easy, as the rule, to find out the relationships in the ideal J_π that "make" the dimension of the quotient algebra finite. Choosing these relationships and factorizing on them, we obtain finite-dimensional "Hecke alike" algebra, generated by π. If the representation π "depends" of some parameter λ the algebra $H_{\pi,J,n}$ can often be "lifted" to the algebra over the ring $k[\lambda, f(\lambda)^{-1}]$ ($f(\lambda)$) polynimial over the field k), where λ is undeterminate (it sometimes becomes difficult to find that this algebra originates from the representation π). Algebras obtained in this way, it seems, will play similar role in the construction of invariants for knots and links, as it plays algebra Hecke $H(n,q)$ [J] (the most important problem is to find the "trace" with special properties on these algebras).

Let us illustrate the above case when π is from the series II, $k = \mathcal{C}, B = q$, where $q \in \mathcal{C}, q \neq 0$ (it is the Burau representation, see [B]).

We have $\tilde{t}_i = \begin{pmatrix} 0 & q \\ 1 & 1-q \end{pmatrix}$ (see 1.2, 2.1, 2.4(ii)). It is easy to verify that (i) $\tilde{t}_i^2 - (1-q)\tilde{t}_i - q \cdot I = 0$. Representation π defines the map: $\mathcal{C}B_n \to \mathcal{C} < \pi(t_1, ..., \pi(t_{n-1}) >$. Let J_π be the kernel of this map.

Relationship (i) belongs to J_π (and alone "makes" the dimension of the quotient algebra $\mathcal{C}B_n/J_\pi$ finite). Let J be the ideal in $\mathcal{C}B_n$ generated by relationship (i). Then $\mathcal{C}B_n/J = H(q,n)$, where $H(q,n)$ is the algebra Hecke, $\dim_{\mathcal{C}} H(q,n) = n! < \infty$, ($\dim_{\mathcal{C}} \mathcal{C} < \pi(t_1), ..., \pi(t_{n-1}) >< n^2$, hence $J \subset J_\pi$ and $J \neq J_\pi$). The representations of the algebra $H(q,n)$ (that is generated by π) in particular "multiply" the representation π. If q is undeterminate, we can "lift" the algebra $H(q,n)$ to the algebra over the ring $\mathcal{C}[q,q^{-1}]$ of the Laurent polynomials. Algebra that appears plays an important role in the construction of polynomial invariants of knots and links [J], [F–O].

We see, that algebra Hecke $H(q,n)$ is one of the "Hecke alike" algebras, that is generated by the representations of the braid group B_∞, that appears here. We hope to devote our following publication to the study and investigation of these "Hecke alike" algebras (in particular generated by representations from the series I–VI) and invariants of links that possibly could happen here.

2.17. We indicate one case when the invariant of links appears directly (and the "trace" is "isual" trace).

Let π be the finite dimensional representation of the braid algebra $\mathcal{B}$ into $\mathrm{End}_k(V)$ (V is the vector space over the field $k = R$ or $k = \mathcal{C}$), such that linear operator $V \oplus V \to V \oplus V$ with matrix $\begin{pmatrix} \pi(A) & \pi(B) \\ \pi(C) & \pi(D) \end{pmatrix}$ is nondegenerate. Then there exist elements A_1, B_1, C_1, D_1 of the algebra $\mathcal{B}$ such that the following equality holds

$$\begin{pmatrix} \pi(A) & \pi(B) \\ \pi(C) & \pi(D) \end{pmatrix} \begin{pmatrix} \pi(A_1) & \pi(B_1) \\ \pi(C_1) & \pi(D_1) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Note that linear operators $\pi(A_1), \pi(B_1), \pi(C_1), \pi(D_1)$ on the vector space V are uniquely determined by the representation π.

Definition. *We shall call the above representation π of the braid algebra "simple" representation if the following conditions hold:*

(i) $\operatorname{tr}\pi(AX) = \operatorname{tr}\pi(X) + tq_1(X), \quad q_1(X) \in Z;$

(ii) $\operatorname{tr}\pi(A_1X) = \operatorname{tr}\pi(X) + tq_2(X); \quad q_2(X) \in Z$

for some $t \in k, t \neq 0$ and for any $X \in \mathcal{B}$ one of the following forms:

(iii) *X is the product of some elements of algebra $\mathcal{B}$, each equals D or D_1, or $X = 1$;*

(iv) *$X = G_1 S_1 M S_2 G_2$, where $G_1(G_2)$*

is 1 or the product of some elements of algebra $\mathcal{B}$ each equals D or D_1, $S_1(S_2)$ equals C or $C_1(B$ or $B_1)$, M is the product of some elements of algebra $\mathcal{B}$, each equals to one the elements A, A_1, B, B_1, C, C_1, D, D_1, such that $n_1 + n_2 = n_3 + n_4$, where n_1, n_2, n_3, n_4 are the numbers of elements B, B_1, C, C_1 respectively, in this product.

Theorem. *Let π be "simple" representation of the braid algebra $\mathcal{B}$. For any oriented link $\hat{\beta}$, obtained of the braid $\beta \in B_n$ (see [B]), let $I(\hat{\beta}) = e^{\frac{\pi i}{t}[2\mathrm{tr}\,\pi'(\beta) + (\exp\beta)\mathrm{tr}\,[\pi(D_1) - \pi(D)] - n\mathrm{tr}\,[\pi(D_1) + \pi(D)]]}$ where π' is the representation of the group B_n into $\mathrm{End}_k\left(\bigoplus\limits_{i=1}^{n} V\right)$ obtained of the representation π (theorem 2.2),* $\exp\beta = \alpha_1 + \cdots + \alpha_k$*, if $\beta = t_{i_1}^{\alpha_1} ... t_{i_k}^{\alpha_k}$. Then $I(\hat{\beta})$ is the invariant of the oriented link $\hat{\beta}$.*

Scetch of the proof. It is obvious that $I(\hat{\beta})$ does not depend of Markov move of type 1 (see [B]). Besides this, it does not depend of Markov move of type 2 (see [B]) if we take into account, that $\mathrm{tr}\,\pi'(t_n\beta) = \mathrm{tr}\,(\beta) + \mathrm{tr}\,D + tq_1(X_1)$, $\mathrm{tr}\,\pi'(t_n^{-1}\beta) = \mathrm{tr}\,\beta + \mathrm{tr}\,D_1 + tq_2(X_2)$, where $X_1(X_2)$ is the sum of some elements of algebra $\mathcal{B}$ each of the form that is indicated in definition, $q_1(X_1)(q_2(X_2))$ is the corresponding sum of the values of the function $q_1(q_2)$ on these summands. Thus, according to Markov theorem (see [B]), $I(\hat{\beta})$ is invariant of the link $\hat{\beta}$.

Remark. *The theorem remains true if we shall demand the holding of the relations* (i), (ii) *for the X of the form* (iii), (iv)*, and such that $\psi(i) = n_1(i) + n_2(i) - n_3(i) - n_4(i) \geq 0$ for any $1 \leq i < l(X)$, where $n_1(i)$, $n_2(i)$, $n_3(i)$, $n_4(i)$ are the numbers of appeareces of generators B, B_1, C, C_1, respectively, in monom X, from the first position of the monom to the i^{th} position, $l(X)$ — the length of the monom X. See also 3.9, 3.10, 3.11.*

3 Some invariants of oriented knots and links

In this part of the paper we indicate new methods of constructing invariants of knots and links (different from "classical" method, see item 2.16 and [J]). We find some invariants of oriented knots and links.

3.1. **Remark.** *There exists well known geometric realization of braid group B_n, (see [B]). For geometric braids $\beta_1, \beta_2 \in B_n$, $\beta_1 \cdot \beta_2$ is the result of concatenation of braids β_1, β_2 (β_2 – first).*

3.2. **Definition.** *Let G be any group. By the G-braid of n strings we shall understand any permutation π on the set $\{1, 2, ..., n\}$, with the map, that to any pair $(i, \pi(i))$ $i \in N, i \leq n$ corresponds some element $g_i \in G$.*

It is obvious what way we have to multiply two G-braids of n-strings and that we obtain the group. Let denote it $B_n(G)$. It is obvious, that $B_n(G) \subset B_{n+1}(G)$, so we can consider the group $B_\infty(G) = \underset{n}{\cup} B_n(G)$.

3.3. Let a_i, b_i, $i = 1, \infty$ be the sequence of elements of the group G, such that the following relations hold:

$$\text{(i)} \quad a_i b_i = b_{i+1} a_{i+1} \qquad \text{for any} \quad i \in N$$

(these relations appeared first in [M]).

Let us correspond to any $t_i \in B_\infty$ the G-braid in $B_\infty(G)$, such that $\pi(i) = i + 1$, $\pi(i + 1) = i$, $\pi(k) = k$, if $k \notin \{i, i + 1\}$, $g_i = a_i$, $g_{i+1} = b_i$, $g_k = 1$, if $k \notin \{i, i + 1\}$. We shall denote this G-braid $\tilde{t}_i$.

Theorem. *The map $t_i \to \tilde{t}_i$ defines homomorphism $B_\infty \to B_\infty(G)$.*

Proof. It is easy to verify that it is equivalent to the relations (i).

The image of the braid $\beta \in B_\infty$ under this homomorfism we shall denote by $\tilde{\beta}(\tilde{\beta} \in B_\infty(G))$. We see, that to every string of the braid β corresponds some element of the group G.

3.4. Given $\beta \in B_n$, one can correspond a braid in the space and may form an oriented link $\hat{\beta}$ by identifying the point j at the top and the point j in the bottom ($j = 1, ..., n$) (see [B]). Any oriented link arises in this way, and the question of when two braids give rise to isotopic links is answered by

Theorem. (A.A. Markov, see [B]). *Let B be the disjoint union*

$$B = \bigsqcup_{n \geq 1} B_n \quad (B_1 = \{1\}).$$

Define the equivalence relation on B as the one which is generated by the relations

(i) *If β and $\gamma \in B_n$, then $\beta \equiv \gamma\beta\gamma^{-1}$,*
(ii) *If $\beta \in B_n$, then $\beta \equiv t_n\beta$ and*
$\beta \equiv t_n^{-1}\beta$ *(where $t_n^{\pm 1}\beta \in B_{n+1}$).*

Then two braids β, β' give rise to isotopic links if and only if $\beta \equiv \beta'$.

3.5. Next simple theorem gives the key to this part of the paper.

Theorem. *Let G be any group and a_i, b_i $i \in N$ be any sequence of its elements that satisfy the relations 1.3 (i). For $\beta \in B_n$ let us go from any point $j \in N$ (in the top or in the bottom) of some knot in $\hat{\beta}$ in the direction of this knot multiplying subseguently corresponding strings in $\tilde{\beta} \in B_n(G)$, that are the elements of G, (the stick of the point j in the top with the point j in the bottom corresponds to $1 \in G$) until we return to the starting point. Let us correspond to the every knot in $\hat{\beta}$ the conjugacy class of the result of this multiplication. Then if $b_i = a_i^{-1}$, the family of this cojugacy classes of the group G (their number is the number of knots in $\hat{\beta}$) is invariant of the link $\hat{\beta}$.*

Proof. Let $\beta_2 \in B_n$ be obtained from $\beta_1 \in B_n$ by Markov move 3.4 (i). Then $\beta_2 = \gamma\beta_1\gamma^{-1}$ for some $\gamma \in B_n$ and, hence, $\tilde{\beta}_2 = \tilde{\gamma}\tilde{\beta}_1\tilde{\gamma}^{-1}$.

Let us go from any point of some knot in $\hat{\beta}_2$ in the direction of orientation of link β_2 multiplying subsequently elements of group G corresponding to the strings, that we are passing through, until we return to the starting point. It is not difficult to see, taking into account the equality $\tilde{\beta}_2 = \tilde{\gamma}\tilde{\beta}_1\tilde{\gamma}^{-1}$, that the conjugacy class of the result of multiplication is equal to the conjugacy class for the corresponding knot in $\hat{\beta}_1$. Thus the families of conjugasy classes so obtained for link $\hat{\beta}_1$ and for link $\hat{\beta}_2$, coincide.

Let $\beta_1 \in B_n$ and $\beta_2 = t_n^{\pm}\beta_1$, i.e. β_2 is obtained of β_1 by Markov move 3.4 (ii)). Then, evidently, the only one conjugacy class for $\hat{\beta}_2$ could changes (according to those in $\hat{\beta}_1$), namely the conjugacy class that corresponds to the knot passing through the $(n+1)^{\text{th}}$ point in the

bottom. But it does not change because the only new multiplier that appears in corresponding product is element $b_n a_n = 1$. Thus the family of conjugacy classes does not depend on Markov moves, that proves the theorem.

Consequence. *For the field k and any $m \in N$ let $K = k(\{a_{ij}^{(s)}\})$ be the field of the rational functions of the undeterminates $\{a_{ij}^{(s)}\}, i, j = 1, ..., m;$ $s \in N$ over the field k (every function depends of the finite number of undeterminates). For the sequence $a_s = (a_{ij}^{(s)})$, $b_s = (a_{ij}^{(s)})^{-1}$, $s = 1, \infty$ of the elements of $GL_m(K)$ and every link $\hat{\beta}(\beta \in B_n$ for some $n \in N)$ consider the family of characteristic polynomials of any integer power t of representatives of the conjugacy classes, indicated in the theorem. Then this family of polynomials of X in $k[\{a_{ij}^{(s)}\}, \frac{1}{\det(a_{ij}^{(s)})}, X]$ is invariant of the link.*

Proof. It is obvious.

Remark. *Every polynomial pointed out in the consequence has the degree m, as the polynomial of X.*

3.6. The sequence $a_i, b_i = a_i^{-1}$, $i = 1, \infty$, $a_i \in G$ of the theorem 3.5 gives the map $B_\infty \to B_\infty(G)$, such that $\tilde{t}_i^2 = 1$. Thus the invariants of the theorem 3.5 (and the consequence 3.5) do not differ links $\hat{\beta}_1$ and $\hat{\beta}_2$ if the permutations that define the braids β_1 and β_2 coincide (despite this, they may give some useful information, for instance, about the braid index of the link, especially in the case when G is a free group freely generated by the elements a_i).

To avoid this, we have to consider the case, when $a_i b_i \neq 1$.

3.7. **Theorem.** *For the field k and any $m \in N$ let $K = k(\{a_{ij}^{(s)}\})$ be the field of the rational functions of the undeterminates $\{a_{ij}^{(s)}\}$, $i, j = 1, ..., m$, $s \in N$ over the field k. Let $K[T, T^{-1}]$ be the ring of Laurent polynomials of the undeterminate T over the field K. For the sequence $a_s = (a_{ij}^{(s)})$, $b_s = T \cdot (a_{ij}^{(s)})^{-1}$ of the elements of $GL_m(K[T, T^{-1}])$ and every link $\hat{\beta}$ ($\beta \in B_n$ for some $n \in N$) consider the family of characteristic polynomials of any integer power t of representatives of the conjugacy*

classes, indicated in the theorem 3.5. For $0 \le l < m$ *let us consider the product* $P_l = T^{t(l-m)\exp\beta}$. $P_1^{(l)} ... P_r^{(l)}$, *where* r *is the number of knots in link* $\hat{\beta}$, $P_1^{(l)}, ..., P_r^{(l)}$ *are the multipliers nearby* X^l *of these characteristic polynomials,* $\exp\beta$ *is the exponent sum of* β ($\exp\beta = \alpha_1 + \cdots + \alpha_p$ *if* $\beta = t_{i_1}^{\alpha_1} ... t_{i_p}^{\alpha_p}$). *Then the sequence* $P_0, ..., P_{m-1}$ *of elements in* $K[T, T^{-1}]$ *is invariant of the link* $\hat{\beta}$.

Proof. The fact that the sequence $P_0, ..., P_{m-1}$ does not depend of the Markov move 3.4 (i) is obvious. If we have the Markov move 3.4 (ii), then one and only one of conjugacy classes in $GL_m(K[T, T^{-1}])$ that are given in theorem 3.5 multiplies on T if the move is $\beta \equiv t_n \beta$, or it multiplies on T^{-1}, if the move is $\beta \equiv t_n^{-1} \beta$. Taking into account that $\exp(t_n\beta) = \exp\beta + 1$, $\exp(t_n^{-1}\beta) = \exp\beta - 1$ we obtain that the sequence $P_0, ..., P_{m-1}$ does not depend of Markov moves and thus it is invariant of the link $\hat{\beta}$.

Remark. *Let* $\hat{\beta}$ *be a link. Any* $m \in N$ *and any integer* t *define sequence* $P_0, ..., P_{m-1}$, *and* $P_i \in k\left[\{a_{ij}^{(s)}\}, \frac{1}{\det(a_{ij}^{(s)})}, X, T, T^{-1}\right]$.

3.8. **Theorem.** *Let* G *be the group and* $Tr : G \to k$ *be the map from* G *into the field* k *such that* $Tr(g^{-1}xg) = Tr(x)$ *for any* $g, x \in G$. *Suppose that there exists an element* $u \in G$ *such that* $Tr(ux) = \lambda_1 Tr(x)$, $Tr(u^{-1}x) = \lambda_2 Tr(x)$, *for any* $x \in G$, *where* $\lambda_1, \lambda_2 \in k$, $\lambda_1\lambda_2 \neq 0$. *Then any sequence* $a_s \in G$, $s \in N$ *of elements of the group* G *defines invariant of any link* $\hat{\beta}$ ($\beta \in B_n$ *for some* n) *obtained in the following way. For the sequence* $a_s, b_s = a_{s-1} ... a_1 u a_1^{-1} ... a_s^{-1} a_s^{-1}$ $s = 1, 2...$ *(this sequence satisfies the relations 3.3 (i)) we consider the representatives* $\gamma_1, ..., \gamma_r$ *of conjugacy classes for* $\hat{\beta}$, *that is given by the theorem 3.5. Let*

$$\text{(i)} \quad I_{\hat{\beta}} = k_n \cdot V^{\exp\beta} \cdot \prod_{i=1}^{r} Tr(\gamma_i)\,,$$

where r *is the number of knots in link* $\hat{\beta}$,

$$V^2 = \lambda_2 \cdot \lambda_1^{-1}, \quad k_1 = 1, \; k_{n+1} = k_n \cdot V^{-1} \cdot \lambda_1^{-1}.$$

Then element $I_{\hat{\beta}} \in k\left(\sqrt{\frac{\lambda_2}{\lambda_1}}\right)$ *is invariant of the link* $\hat{\beta}$.

Proof. It is analogous to the proof of the theorem 3.7. The only difference is that we consider invariant in the form (i). The condition

$I_{\widehat{t_n\beta}} = I_{\widehat{t_n^{-1}\beta}}$ implies equality $\lambda_1 V \prod_{i=1}^{r} Tr(\gamma_i) = \lambda_2 V^{-1} \prod_{i=1}^{r} Tr(\gamma_1)$ and hence $V^2 = \lambda_2 \lambda_1^{-1}$. The condition $I_{\hat{\beta}} = I_{\widehat{t_n\beta}}$ implies equality

$$k_n \cdot \prod_{i=1}^{r} Tr(\gamma_i) = k_{n+1} \cdot V \cdot \lambda_1 \cdot \prod_{i=1}^{r} Tr(\gamma_i)$$

that implies relation $k_{n+1} = k_n \cdot V^{-1} \cdot \lambda_1^{-1}$.

Remark. *In fact one can easily prove that if* $Tr \not\equiv 0$*, then* $\lambda_2 = \lambda_1^{-1}$ *and hence* $V = \pm\lambda_1^{-1}$*.*

In the followings items we give another one approach to the construction of invariants of oriented knots and links.

3.9. Let $\overline{\mathcal{B}}$ be an algebra over the ring Z with generators $A, B, C, D, A_1, B_1, C_1, D_1$ and the following defining relations:

(i) $A^2 + BAC = A$,
(ii) $D^2 + CDB = D$,
(iii) $CB - BC = ADA - DAD$,
(iv) $BA - AB = BAD$,
(v) $AC - CA = DAC$,
(vi) $DB - BD = ADB$,
(vii) $CD - DC = CDA$,
(viii) $AA_1 + BC_1 = 1$,
(ix) $CA_1 + DC_1 = 0$,
(x) $AB_1 + BD_1 = 0$,
(xi) $CB_1 + DD_1 = 1$.

Then $\overline{\mathcal{B}} = \mathcal{B}\langle A_1, B_1, C_1, D_1\rangle$, where $\mathcal{B}$ is the braid algebra (if we replace in the definition of the braid algebra the field k on the ring Z).

Note that in the group $GL_2(\overline{\mathcal{B}})$ matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is invertible and

$$\text{(xii)} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.$$

It is the main reason to introduce the generators A_1, B_1, C_1, D_1.

Let X be formal "noncommutative monom" of the elements $A, A_1, B, B_1, C, C_1, D, D_1$ that is the product of some of the elements $A, A_1, B, B_1, C, C_1, D, D_1$ of the algebra $\overline{\mathcal{B}}$ in some order (for instance $X = A_1^3\, B_1\, A_1\, B^2\, C_1^5\, D\, C_1$). Let $l(X)$ be the length of this monom. For $1 \le i \le l(X)$ let $\psi(i) = n_1(i) + n_2(i) - n_3(i) - n_4(i)$, where $n_1(i), n_2(i), n_3(i), n_4(i)$ are the numbers of the appearences of B, B_1, C, C_1, respectively in the part of this monom from the first position to the i^{th} position (for the

above example $n_1(4) = 0, n_2(4) = 1, n_3(4) = 0, n_4(4) = 0, n_1(12) = 2$, $n_2(12) = 1, n_3(12) = 0, n_4(12) = 6$).

Definition. *We shall call the subgroup in additive group $\overline{\mathcal{B}}$ generated by $1 \in \overline{\mathcal{B}}$ and monoms X (after real multiplication), for which $\psi(l(X)) = 0$ ($\psi(l(X)) = 0$ and $\psi(i) \geq 0$, for any $1 \leq i \leq l(X)$) the group T_{nc} of diagonal elements (the group T_{ncl} of the diagonal elements in the last position of the diagonal).*

Note that in reality both of this subgroups are subrings of the ring $\overline{\mathcal{B}}$. We will denote by J the right ideal of ring T_{ncl} generated by elements $A-1, A_1-1$. Thus $J = (A-1)T_{ncl} + (A_i - 1)T_{ncl}$ and $J \subset T_{ncl} \subset Tnc$. For any $k \in N$ let

$$\{a_{ij}\}_{\substack{i=1,k \\ j=1,k}}, \quad \{b_{ij}\}_{\substack{i=1,k \\ j=1,k}}, \quad \{c_{ij}\}_{\substack{i=1,k \\ j=1,k}}, \quad \{d_{ij}\}_{\substack{i=1,k \\ j=1,k}}$$

be $4k^2$ undeterminates. Let $P = Z[\{a_{ij}\}, \{b_{ij}\}, \{c_{ij}\}, \{d_{ij}\}, u]$ be the commutative algebra over Z (that is the ring) generated by these $4k^2$ undeterminates and undeterminate u, with following defining relation: $u \times \det \begin{pmatrix} \varphi(A) & \varphi(B) \\ \varphi(C) & \varphi(D) \end{pmatrix} = 1$ plus $7k^2$ relations that come from relations (i)-(vii) if we rely

$$\varphi(A) = (a_{ij})_{k\times k}, \quad \varphi(B) = (b_{ij})_{k\times k},$$

$$\varphi(C) = (c_{ij})_{k\times k}, \quad \varphi(D) = (d_{ij})_{k\times k}$$

and substitute in the relations (i)-(vii) $\varphi(A), \varphi(B), \varphi(C), \varphi(D)$ instead of A, B, C, D respectively (thus the algebra P gives the "common point" of those k-dimensional representations of the braid algebra $\mathcal{B}$, that lead to the representations of the braid group B_n for any $n \in N$ according to the theorem 2.2). Matrix $\begin{pmatrix} \varphi(A) & \varphi(B) \\ \varphi(C) & \varphi(D) \end{pmatrix}$ over the ring P is invertible . Let us define $\varphi(A_1)$, $\varphi(B_1)$, $\varphi(C_1)$, $\varphi(D_1)$ by the equality $\begin{pmatrix} \varphi(A_1) & \varphi(B_1) \\ \varphi(C_1) & \varphi(D_1) \end{pmatrix} = \begin{pmatrix} \varphi(A) & \varphi(B) \\ \varphi(C) & \varphi(D) \end{pmatrix}^{-1}$ (i.e. matrices A_1, B_1, C_1, D_1 become concrete matrices with entries from the ring P). Thus we have obvious homomorphism $\varphi : \overline{\mathcal{B}} \to \mathrm{Mat}_{k\times k}(P)$ and map $\mathrm{tr}\,\varphi : \overline{\mathcal{B}} \to P$ that to every element $b \in \overline{\mathcal{B}}$, corresponds $\mathrm{tr}\,\varphi(b)$ (where trace is the usual trace of the matrix, i.e. the sum of its diagonal elements). Note that

$\operatorname{tr}\varphi(T_{nc})$, $\operatorname{tr}\varphi(T_{ncl})$, $\operatorname{tr}\varphi(J)$ are subgroups of additive group of the ring P and $\operatorname{tr}\varphi(J) \subset \operatorname{tr}\varphi(T_{ncl}) \subset \operatorname{tr}\varphi(T_{nc})$.

Definition. *We shall call the subgroup in $L_k^{\text{link}}(L_k^{\text{d.link}}))$ additive group P that is the image $\operatorname{tr}\varphi T_{nc}(\operatorname{tr}\varphi(J))$ of the group T_{nc} (of the group J) under this map $\operatorname{tr}\varphi$ — the k^{th} group of links (the k^{th} group of degenerate links) and the quotient group $L_k^{\text{inv}} = L_k^{\text{link}}/L_k^{\text{d.link}}$ — the k^{th}-group of invariants of links.*

3.10. For any $n \in N$ let us consider the homomorphism π of the braid group $B_n = \langle t_1, ..., t_{n-1}\rangle$ in the group $GL_{nk}(P)$, such that

$$\pi(t_i) = \begin{pmatrix} I_{(i-1)k\times(i-1)k} & 0 & 0 \\ 0 & \begin{matrix} \varphi(A) & \varphi(B) \\ \varphi(C) & \varphi(D) \end{matrix} & 0 \\ 0 & 0 & I_{(n-i)k\times(n-i)k} \end{pmatrix},$$

According to the items 2.2, 3.9 such homomorphism exists and it is obviously unique.

Lemma . *For any $\beta \in B_n$ the trace of the matrix $\pi(\beta)$ belongs to the group L_k^{link}, and $k \times k$ matrix in the right low corner of the matrix $\pi(\beta)$ belongs to the subring $\varphi(T_{ncl})$ of the ring $\operatorname{Mat}_{k\times k}(P)$ π.*

Proof. Let us consider in $kn \times kn$ matrix $\pi(\beta)$ n square $k \times k$ submatrices each with left upper vertix laying on the diagonal of matrix $\pi(\beta)$ and which do not intersect. Then from the definition of the representation π every of this matrices belongs to the $\varphi(T_{nc})$ and the $k \times k$ matrix in the right low corner of matrix $\pi(\beta)$ belongs to $\varphi(T_{ncl})$.

For any commutative ring $\mathcal{A}$ and any $g, g_1 \in GL_r(\mathcal{A})$ $\operatorname{tr}(g_1 g g_1^{-1}) = \operatorname{tr} g$ (if in the case when $\mathcal{A}$ is a field it is well known fact, the common case reduces to this one if we consider the "abstract" matrices $\tilde{g}_1 = (x_{ij})_{r\times r}$, $\tilde{g} = (y_{ij})_{r\times r}$ over the field $Q(\{x_{ij}\}, \{y_{ij}\}$, where x_{ij}, y_{ij} $i, j = 1, ..., r$ are undeterminates).

3.11. For the element $l \in L_k^{\text{link}}$ by $l^\wedge$ we denote the element $f(l) \in L_k^{\text{inv}}$,

where f is the natural homomorphism $L_k^{\text{link}} \to L_k^{\text{link}}/L_k^{\text{d.link}} = L_k^{\text{inv}}$ (see definition 3.9).

Theorem. *Let $\hat{\beta}$ be the oriented link obtained of the braid $\beta \in B_n$. Then the element*
$I_{\beta,k} = [2\mathrm{tr}\,\pi(\beta) + (\exp\beta)(\mathrm{tr}\,\varphi(D_1) - \mathrm{tr}\,\varphi(D)) - n(\mathrm{tr}\,\varphi(D_1) + \mathrm{tr}\,\varphi(\mathrm{D}))]^\wedge$
(where π is standart representation of B_n into $GL_{nk}(P)$, see 3.10, φ is standart homomorphism $\overline{\mathcal{B}} \to \mathrm{Mat}_{k\times k}(P)$, see 3.9) of the group L_k^{inv} is invariant of the link $\hat{\beta}$.

Proof. If $\beta_1, \beta_2, \gamma \in \mathrm{B}_n$ and $\beta_2 = \gamma\beta_1\gamma^{-1}$ then $\pi(\beta_2) = \pi(\gamma)\pi(\beta_1)$ $\pi(\gamma^{-1}$ and, hence, $\mathrm{tr}\,\pi(\beta_2) = \mathrm{tr}\,\pi(\beta_1)$. So $I_{\beta,k}$ does not depend of the Markov move 3.4 (i). Let $\beta_1 \in \mathcal{B}_n$, $\beta_2 \in \mathcal{B}_{n+1}$ and $\beta_2 = t_n^\delta \beta_1$, where $\delta = 1$, or $\delta = -1$. Let us consider the diagonal elements of the matrices $\pi(\beta_1) \in GL_{nk}(P)$ and $\pi(\beta_2) \in GL_{(n+1)k}(P)$. As follows from the definition of the representation π first $(n-1)k$ elements of the diagonal of matrices $\pi(\beta_1)$ and $\pi(\beta_2)$ coincide, $k \times k$ matrix R in the right low corner of matrix $\pi(\beta_1)$ belongs to the algebra $\varphi(T_{ncl})$ (see lemma 3.10). The corresponding $k\times k$ matrix of matrix $\pi(t_n^\delta\beta_1)$ (we mean $k\times k$ matrix in matrix $\pi(t_n^\delta\beta_1)$ coordinates of vertices of which coincide with coordinates of matrix R in matrix $\pi(\beta_1)$) equals $\varphi(A)\cdot R$ if $\delta = 1$, and $\varphi(A_1)\cdot R$ if $\delta = -1$. The $k \times k$ matrix in the right low corner of the matrix $\pi(t_n^\delta\beta_1)$ equals $\varphi(D)$, if $\delta = 1$, and $\varphi(D_1)$ if $\delta = -1$. We have

$$
\begin{aligned}
&I_{t_n\beta_1,k} = [2\mathrm{tr}\,\pi(t_n\beta_1) + (\mathrm{tr}\,\varphi(D_1) - \mathrm{tr}\,\varphi(D))(\exp t_n\beta_1) - \\
&-(n+1)(\mathrm{tr}\,\varphi(D_1) + \mathrm{tr}\,\varphi(D))]^\wedge = \\
&= [2\mathrm{tr}\,\pi(\beta_1) + 2\mathrm{tr}\,(\varphi(A-1))R + 2\mathrm{tr}\,\varphi(D) + \\
&+(\mathrm{tr}\,\varphi(D_1) - \mathrm{tr}\,\varphi(D))(\exp\beta_1 + 1) - \\
&-(n+1)(\mathrm{tr}\,\varphi(D_1) + \mathrm{tr}\,\varphi(D))]^\wedge = \\
&= [2\mathrm{tr}\pi(\beta_1) + (\mathrm{tr}\varphi(D_1) - \mathrm{tr}\varphi(D))\exp\beta_1 - n(\mathrm{tr}\varphi(D_1) + \mathrm{tr}\varphi(D)) + \\
&+\mathrm{tr}\,\varphi(D_1) - \mathrm{tr}\,\varphi(D) - \mathrm{tr}\,\varphi(D_1) - \\
&-\mathrm{tr}\,\varphi(D) + 2\mathrm{tr}\,\varphi(D) + 2\mathrm{tr}\,((\varphi(A-1))R)]^\wedge = I_{\beta_1,k}
\end{aligned}
$$

(we took into account that $\exp t_n\beta_1 = \exp\beta_1+1$ and that $2(\varphi(A-1))R \in \varphi(J_)$ and, hence, $2\mathrm{tr}\,((\varphi(A-1))R) \in L_n^{\mathrm{d.link}}$). We have

$$I_{t_n^{-1}\beta_1} = [2\mathrm{tr}\,\pi(t_n^{-1}\beta_1) + (\mathrm{tr}\,\varphi(D_1) - \mathrm{tr}\,\varphi(D))\,(\exp\beta_1 - 1) -$$

$$-(n+1)(\mathrm{tr}\,\varphi(D_1) + \mathrm{tr}\,\varphi(D))]^\wedge =$$

$$= [2\mathrm{tr}\,\pi(\beta_1) + 2\mathrm{tr}\,((\varphi(A_1-1))R) + 2\mathrm{tr}\,\varphi(D_1) +$$

$$+(\mathrm{tr}\,\varphi(D_1) - \mathrm{tr}\,\varphi(D))(\exp\beta_1 - 1) - (n+1)(\mathrm{tr}\,\varphi(D_1) + \mathrm{tr}\,\varphi(D))]^\wedge =$$

$$= [2\mathrm{tr}\,\pi(\beta_1) + (\mathrm{tr}\,\varphi(D_1) - \mathrm{tr}\,\varphi(D))\exp\beta_1 - n(\mathrm{tr}\,\varphi(D_1) + \mathrm{tr}\,\varphi(D)) -$$

$$-\mathrm{tr}\,\varphi(D_1) + \mathrm{tr}\,\varphi(D) - \mathrm{tr}\,\varphi(D_1) - \mathrm{tr}\,\varphi(D) +$$

$$+2\mathrm{tr}\,\varphi(D_1) + 2\mathrm{tr}\,((\varphi(A_1-1))R)]^\wedge = I_{\beta_1,k}$$

(we took into account that $\exp t_n^{-1}\beta_1 = \exp\beta_1 - 1$ and that $2\varphi(A_1-1)R \in J$. Thus $I_{\beta_2,k} = I_{\beta_1,k}$ and, hence, $I_{\beta,k}$ does not depend of Markov move 3.4 (ii). So, according to Markov theorem 3.4, $I_{\beta,k}$ is invariant of the link $\hat\beta$.

There is the hope that group L_k^{inv} (or its suitable quotient group) can be calculated at least for small k. For $k=1$, it easy to calculate, that $L_1^{\mathrm{inv}} = Z$.

Note that similar theorem can be easily formulated and proved for the case $(n,1)$. It is more complex to formulate the analogous theorem for the case (n,λ), where $\lambda > 2$, see 1.1, 2.14.

3.12. The next theorem and its consequence gives, seems, a very mighty instrument for the construction of the invariants of oriented knots and links.

Theorem. *Let $T(i)$, $i \in N$ be the sequence of tensors that satisfy the condition of the theorem 2.1. Let for any $i \in N$ the following matrices equalities hold $(\gamma(i)^{i_1}{}_{i_2}) = \lambda_1(i) \times I$, $(\tau(i)^{i_1}{}_{i_2}) = \lambda_2(i) \times I$, $\lambda_1(i) \in k$, $\lambda_2(i) \in k$, $\lambda_1(i) \neq 0, \lambda_2(i) \neq 0$, where*

$$\gamma(i)^{i_1}{}_{i_2} = \sum_{j=1}^{\dim V_{i+1}} T(i)^{i_1 j}{}_{i_2 j}, \quad \tau(i)^{i_1}{}_{i_2} = \sum_{j=1}^{\dim V_{i+1}} T^{-1}(i)^{i_1 j}{}_{i_2 j}$$

($(T^{-1}(i)^{i_1 i_2}{}_{j_1 j_2})$ is matrix inverse to the matrix $(T(i)^{i_1 i_2}{}_{j_1 j_2})$. If $\lambda_2(i)\cdot \lambda_1^{-1}(i) = \mathrm{const}$ (i.e. $\lambda_2(i)\lambda_1^{-1}(i)$ does not depend of i) then for any link $\hat{\beta}, \beta \in B_n$ element $I_\beta = k_n V^{\exp \beta} \mathrm{Tr}\,(\beta) \in k\left(\sqrt{\lambda_2(1)\lambda_1^{-1}(1)}\right)$ where $V = \sqrt{\lambda_2(1)\lambda_1^{-1}(1)}, k_1 = 1,\ k_{n+1} = k_n V^{-1}(\lambda_1(n))^{-1}$ is invariant of the link $\hat{\beta}$ (π is the representation of the group B_n determined by the tensor´s sequence $T(i)$, see theorem 2.1).

Proof. Let $\beta_1 \in B_n$ be obtained from $\beta \in B_n$ by Markov move 3.4 (i). Then $\beta_1 = \gamma\beta\gamma^{-1}$ for some $\gamma \in B_n$ and, hence, $I_{\beta_1} = I_\beta$.

Let $\beta_1 \in B_{n+1}$ be obtained from β by Markov move 3.4 (ii). Then $\beta_1 = \beta t_n^\delta$, where $\delta = \pm 1$. As $\pi(\beta) \in GL_k(V_1 \otimes \cdots \otimes V_n)$ we have

$$\pi(\beta) = \sum T^{i_1 \ldots i_n}{}_{j_1 \ldots j_n} e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n} \otimes e^{(1) j_1 *} \otimes \cdots \otimes e^{(n) j_n *}.$$

Then $\mathrm{tr}\,\pi(\beta) = \sum\limits_{i_1,\ldots,i_n} T^{i_1,\ldots,i_n}{}_{i_1,\ldots,i_n}$ and $\mathrm{tr}\,\pi(\beta t_n^\delta) =$

$$= \sum_{i_1,\ldots,i_n,j} T^{i_1,\ldots,i_n}{}_{i_1,\ldots,i_n} \times T^\delta(n)^{i_n j}{}_{i_n j} +$$

$$+ \sum_{i_1,\ldots,i_n,j,i'_n \neq i_n} T^{i_1,\ldots,i_{n-1} i_n}{}_{i_1,\ldots,i_{n-1} i'_n} t^\delta(n)^{i'_n j}{}_{i_n j} =$$

$$= \sum_{i_1,\ldots,i_n} \left(T^{i_1,\ldots,i_n}{}_{i_1,\ldots,i_n} \sum_j T^\delta(n)^{i_n j}{}_{i_n j} \right) +$$

$$+ \sum_{i_1,\ldots,i_n,i'_n \neq i_n} \left(T^{i_1,\ldots,i_{n-1} i_n}{}_{i_1,\ldots,i_{n-1} i'_n} \sum_j T^\delta(n)^{i'_n j}{}_{i_n j} \right)$$

(here $T^\delta(n)^{i_1 j_1}{}_{i_2 j_2} = T(n)^{i_1 j_1}{}_{i_2 j_2}$ if $\delta = 1$ and $T^{-1}(i)^{i_1 i_2}{}_{j_1 j_2}$ if $\delta = -1$). Thus $\mathrm{tr}\,\pi(\beta t_n) = \lambda_1(n)\mathrm{Tr}\,(\beta)$ and $\mathrm{tr}\,\pi(\beta t_n^{-1}) = \lambda_2(n)\mathrm{Tr}\,(\beta)$. Now it is an easy exercise to show that $I_{\beta t_n} = I_{\beta t_n^{-1}} = I_\beta$ and, hence, I_β does not depend of Markov move 3.4 (i)-(ii). According to Markov theorem I_β is invariant of the link $\hat{\beta}$.

Consequence. *Let V be a finite dimensional vector space over the field k, $\dim V = n$. Let $T \in V^{\otimes 2} \otimes (V^*)^{\otimes 2}$ be tensor that satisfies the conditions of the theorem 2.2 (i.e. it satisfies braid equation and its*

matrix $(T^{i_1i_2}{}_{j_1j_2})$ *is nondegenerate). Suppose that* $(\gamma^{i_1}{}_{i_2})_{n\times n} = \alpha_1 I_{n\times n}$, $(\tau^{i_1}{}_{i_2})_{n\times n} = \alpha_2 I_{n\times n}$, $\alpha_1, \alpha_2 \in k$, $\alpha_1\alpha_2 \neq 0$, *where*

$$\gamma^{i_1}{}_{i_2} = \sum_{j=1}^{n} T^{i_1 j}{}_{i_2 j}, \qquad \tau^{i_1}{}_{i_2} = \sum_{j=1}^{n} (T^{-1})^{i_1 j}{}_{i_2 j}$$

($((T^{-1})^{i_1i_2}{}_{j_1j_2})$ *is matrix inverse to the matrix* $(T^{i_1i_2}{}_{j_1j_2})$*). Then for any link* $\hat{\beta}$, $\beta \in B_n$ *element* $I_\beta = k_n V^{\exp\beta} \mathrm{Tr}\,\pi(\beta) \in k\left[\sqrt{\alpha_2\alpha_1^{-1}}\right]$ *where* $V = \sqrt{\alpha_2\alpha_1^{-1}}$, $k_1 = 1$, $k_{n+1} = k_n V^{-1}\alpha_1^{-1}$ *is invariant of the link* $\hat{\beta}$ *(*π *is the representation of the group* B_n *determined by tensor* T*, see theorem 2.2).*

Proof. It is obvious.

Note that we can consider commutative algebra $k[\{T^{i_1i_2}{}_{j_1j_2}\}, \alpha_1^{\pm 1}, \alpha_2, x, u]$ over the field k with generators $\{T^{i_1i_2}{}_{j_1j_2}\}$, α_1, α_2, x, u and defining relationships, that follows from the braid equation (see 2.2), conditions of the consequence (for instance, $x^2 = \alpha_2\alpha_1^{-1}$, $u \times \det(T^{i_1i_2}{}_{j_1j_2}) = 1$). Thus we obtain the common point of the tensors that satisfy the conditions of the consequence and universal invariant that belongs to this algebra. But there is no confidence that this formal algebra can be calculated. So a concrete examples of such tensors that satisfy the conditions of the theorem or the consequence are very important.

In the following items we give only one of such examples and, hence, the example of invariant of oriented knots and links.

3.13 Without any doubt (especially keeping in mind the results of the item 2.10) there are a huge amount of the decisions of the equations 2.1 (viii) and 2.2 (viii), such that matrices of the corresponding tensors are nondegenerate (see 2.1, 2.2). Next theorem gives only one example of such decision.

Theorem. *Let* $a_i = (\alpha(i)^{i_1}{}_{i_2})_{n\times n}$, $b_i = (\beta(i)^{i_3}{}_{i_4})_{n\times n}$ *be the sequence of nondegenerate square* $n \times n$ *matrices over the field* k*, such that* $a_i b_i = b_{i+1} a_{i+1}$*, and* V *is a vector space over the field* k, $\dim_k V = n$*. Put* $V_i = V$, $T(i) = \sum T(i)^{i_1i_2}{}_{i_3i_4} e_{i_1} \otimes e_{i_2} \otimes e^{i_3 *} \otimes e^{i_4 *}$*, for any* $i \in N$ *and for some basis* $\{e_s\}$, $s = 1, ..., n$*, where* $T(i)^{i_1i_2}{}_{i_3i_4} = \beta(i)^{i_1}{}_{i_4}\alpha(i)^{i_2}{}_{i_3}$*. Then matrices* $(T(i)^{i_1i_2}{}_{i_3i_4})$ *are nondegenerate and tensors* $T(i)$ *satisfy*

the equation 2.1 (viii) *and thus (according to the theorem 2.1) determine for any $m \in N$ the linear representation π of the group B_m.*

Proof. Simple calculation proves it. The fact that matrices $(T(i)^{i_1 i_2}{}_{i_3 i_4})$ are nondegenerate follows from the fact, that $T(i) \in \operatorname{End} V^{\otimes 2}$, is induced by the linear map $V \times V \to V \times V$ with matrix $\begin{pmatrix} 0 & b_i \\ a_i & 0 \end{pmatrix}$, where a_i and b_i are nondegenerate matrices.

3.14. **Theorem.** *For the field F and any $n \in N$ let $k = F(T, \{a(i)^{i_1}{}_{j_1}\})$ be the field of the rational functions (each depends of finite number of undeterminates) over the field F of the undeterminates T, $\{a(i)^{i_1}{}_{j_1}\}$, $i_1, j_1 = 1, ..., n$, $i \in N$. Then the sequence $a_i = (a(i)^{i_1}{}_{j_1})_{n \times n}$, $b_i = T \times (a(i)^{i_1}{}_{j_1})^{-1}_{n \times n}$ square $n \times n$ matrices over the field k satisfies the condition of the theorem 3.13. The sequence of tensors $T(i)$, constructed according to the theorem 3.13, satisfies the condition of the theorem 3.12 with $\lambda_1(i) = T$, $\lambda_2(i) = T^{-1}$. Thus the sequence of tensors $T(i)$ determines, according to the theorem 3.12, invariant of oriented knots and links. Namely, for any link $\hat{\beta}$, $\beta \in B_n$ the element $I_\beta = (T)^{-\exp \beta} \operatorname{Tr} \pi(\beta)$ of the field k is invariant of the link $\hat{\beta}$.*

Proof. Every statement is evident.

The approach above is considered only for the case case $(n, \lambda) = (2, 1)$ (see 2.1) but it can be easy generalized to the case of arbitrary (n, λ) (especially for "small" n, λ), $n, \lambda \in N$, $\lambda < n$ (see 1.1, 2.14). But it refers to all this paper.

Remark. *In a few days before sending this paper to publication the author revealed the work [A], where the algebra $\mathcal{B}_{2,1}$ appeared also. In this article this algebra is considered mostly of physical point of view and some of its unitary infinite dimensional representation are constructed. The theorem similar to the theorem 2.2 in part, corresponding the algebra $\mathcal{B}_{2,1}$, hovewer, is not formulated, but is indicated.*

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